Spring 2012

EE 330

ENGINEERING ELECTROMAGNETICS

HW 10: *Due Friday 30 March* (last HW before Exam #2)
6.28, 7.4, 7.6, 7.13, 7.14, 7.15, 7.20, 7.22, 7.24, 7.27, 7.28, 7.32, 7.36

Problem 6.28 In free space, the magnetic field is given by

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{36}{r}\cos(6\times10^9 t - kz) \quad \text{(mA/m)}.$$

(a) Determine k.

(b) Determine $\mathbf{E}$.

(c) Determine $\mathbf{J}_\text{d}$.

Solution:

(a) From the given expression, $\omega = 6\times10^9$ (rad/s), and since the medium is free space,

$$k = \frac{\omega}{c} = \frac{6\times10^9}{3\times10^8} = 20 \quad \text{(rad/m)}.$$

(b) Convert $\mathbf{H}$ to phasor:

$$\widetilde{\mathbf{H}} = \hat{\boldsymbol{\phi}}\,\frac{36}{r}\,e^{-jkz} \quad \text{(mA/m)}$$

$$\begin{aligned}
\widetilde{\mathbf{E}} &= \frac{1}{j\omega\varepsilon_0}\,\nabla\times\widetilde{\mathbf{H}} \\
&= \frac{1}{j\omega\varepsilon_0}\left[-\hat{\mathbf{r}}\,\frac{\partial H_\phi}{\partial z} + \hat{\mathbf{z}}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rH_\phi)\right] \\
&= \frac{1}{j\omega\varepsilon_0}\left[-\hat{\mathbf{r}}\,\frac{\partial}{\partial z}\left(\frac{36}{r}\,e^{-jkz}\right) + \frac{\hat{\mathbf{z}}}{r}\,\frac{\partial}{\partial r}(36e^{-jkz})\right] \\
&= \frac{1}{j\omega\varepsilon_0}\left[\hat{\mathbf{r}}\,\frac{j36k}{r}\,e^{-jkz}\right] \\
&= \hat{\mathbf{r}}\,\frac{36k}{\omega\varepsilon_0 r}\,e^{-jkz} = \hat{\mathbf{r}}\,\frac{36\times377}{r}\,e^{-jkz}\times10^{-3} = \hat{\mathbf{r}}\,\frac{13.6}{r}\,e^{-j20z} \quad \text{(V/m)}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} &= \Re\mathfrak{e}[\widetilde{E}e^{j\omega t}] \\
&= \hat{\mathbf{r}}\,\frac{13.6}{r}\cos(6\times10^9 t - 20z) \quad \text{(V/m)}.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{J}_\text{d} &= \varepsilon_0\,\frac{\partial \mathbf{E}}{\partial t} \\
&= \hat{\mathbf{r}}\,\frac{13.6}{r}\,\varepsilon_0\frac{\partial}{\partial t}(\cos(6\times10^9 t - 20z)) \\
&= -\hat{\mathbf{r}}\,\frac{13.6\varepsilon_0\times6\times10^9}{r}\sin(6\times10^9 t - 20z) \quad \text{(A/m}^2\text{)} \\
&= -\hat{\mathbf{r}}\,\frac{0.72}{r}\sin(6\times10^9 t - 20z) \quad \text{(A/m}^2\text{)}.
\end{aligned}$$

Problem 7.4 The electric field of a plane wave propagating in a nonmagnetic material is given by

$$\mathbf{E} = [\hat{\mathbf{y}}\, 3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}}\, 4\cos(\pi \times 10^7 t - 0.2\pi x)] \quad \text{(V/m)}$$

Determine

(a) The wavelength.

(b) ε_r.

(c) $\mathbf{H}$.

Solution:

(a) Since $k = 0.2\pi$,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2\pi} = 10 \text{ m}.$$

(b)

$$u_p = \frac{\omega}{k} = \frac{\pi \times 10^7}{0.2\pi} = 5 \times 10^7 \text{ m/s}.$$

But

$$u_p = \frac{c}{\sqrt{\varepsilon_r}}.$$

Hence,

$$\varepsilon_r = \left(\frac{c}{u_p}\right)^2 = \left(\frac{3 \times 10^8}{5 \times 10^7}\right)^2 = 36.$$

(c)

$$\mathbf{H} = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\eta}\hat{\mathbf{x}} \times [\hat{\mathbf{y}} 3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}} 4\cos(\pi \times 10^7 t - 0.2\pi x)]$$
$$= \hat{\mathbf{z}}\frac{3}{\eta}\sin(\pi \times 10^7 t - 0.2\pi x) - \hat{\mathbf{y}}\frac{4}{\eta}\cos(\pi \times 10^7 t - 0.2\pi x) \quad \text{(A/m)},$$

with

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{6} = 20\pi = 62.83 \quad (\Omega).$$

Problem 7.6 The electric field of a plane wave propagating in a lossless, nonmagnetic, dielectric material with $\varepsilon_r = 2.56$ is given by

$$\mathbf{E} = \hat{\mathbf{y}}\, 20 \cos(6\pi \times 10^9 t - kz) \quad \text{(V/m)}$$

Determine:

(a) f, u_p, λ, k, and η.

(b) The magnetic field $\mathbf{H}$.

Solution:

(a)

$$\begin{aligned}
\omega &= 2\pi f = 6\pi \times 10^9 \text{ rad/s}, \\
f &= 3 \times 10^9 \text{ Hz} = 3 \text{ GHz}, \\
u_p &= \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.56}} = 1.875 \times 10^8 \text{ m/s}, \\
\lambda &= \frac{u_p}{f} = \frac{1.875 \times 10^8}{6 \times 10^9} = 3.12 \text{ cm}, \\
k &= \frac{2\pi}{\lambda} = \frac{2\pi}{3.12 \times 10^{-2}} = 201.4 \text{ rad/m}, \\
\eta &= \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{2.56}} = \frac{377}{1.6} = 235.62\ \Omega.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{H} &= -\hat{\mathbf{x}} \frac{20}{\eta} \cos(6\pi \times 10^9 t - kz) \\
&= -\hat{\mathbf{x}} \frac{20}{235.62} \cos(6\pi \times 10^9 t - 201.4z) \\
&= -\hat{\mathbf{x}}\, 8.49 \times 10^{-2} \cos(6\pi \times 10^9 t - 201.4z) \quad \text{(A/m)}.
\end{aligned}$$

Problem 7.13 Compare the polarization states of each of the following pairs of plane waves:

(a) Wave 1: $\mathbf{E}_1 = \hat{\mathbf{x}}2\cos(\omega t - kz) + \hat{\mathbf{y}}2\sin(\omega t - kz)$.
Wave 2: $\mathbf{E}_2 = \hat{\mathbf{x}}2\cos(\omega t + kz) + \hat{\mathbf{y}}2\sin(\omega t + kz)$.

(b) Wave 1: $\mathbf{E}_1 = \hat{\mathbf{x}}2\cos(\omega t - kz) - \hat{\mathbf{y}}2\sin(\omega t - kz)$.
Wave 2: $\mathbf{E}_2 = \hat{\mathbf{x}}2\cos(\omega t + kz) - \hat{\mathbf{y}}2\sin(\omega t + kz)$.

Solution:

(a)

$$\begin{aligned}
\mathbf{E}_1 &= \hat{\mathbf{x}}2\cos(\omega t - kz) + \hat{\mathbf{y}}2\sin(\omega t - kz) \\
&= \hat{\mathbf{x}}2\cos(\omega t - kz) + \hat{\mathbf{y}}2\cos(\omega t - kz - \pi/2), \\
\widetilde{\mathbf{E}}_1 &= \hat{\mathbf{x}}2e^{-jkz} + \hat{\mathbf{y}}2e^{-jkz}e^{-j\pi/2}, \\
\psi_0 &= \tan^{-1}\left(\frac{ay}{ax}\right) = \tan^{-1}1 = 45^\circ, \\
\delta &= -\pi/2.
\end{aligned}$$

Hence, wave 1 is RHC.

Similarly,

$$\widetilde{\mathbf{E}}_2 = \hat{\mathbf{x}}2e^{jkz} + \hat{\mathbf{y}}2e^{jkz}e^{-j\pi/2}.$$

Wave 2 has the same magnitude and phases as wave 1 except that its direction is along $-\hat{\mathbf{z}}$ instead of $+\hat{\mathbf{z}}$. Hence, the locus of rotation of $\mathbf{E}$ will match the left hand instead of the right hand. Thus, wave 2 is LHC.

(b)

$$\begin{aligned}
\mathbf{E}_1 &= \hat{\mathbf{x}}2\cos(\omega t - kz) - \hat{\mathbf{y}}2\sin(\omega t - kz), \\
\widetilde{\mathbf{E}}_1 &= \hat{\mathbf{x}}2e^{-jkz} + \hat{\mathbf{y}}2e^{-jkz}e^{j\pi/2}.
\end{aligned}$$

Wave 1 is LHC.

$$\widetilde{\mathbf{E}}_2 = \hat{\mathbf{x}}2e^{jkz} + \hat{\mathbf{y}}2e^{jkz}e^{j\pi/2}.$$

Reversal of direction of propagation (relative to wave 1) makes wave 2 RHC.

Problem 7.14 Plot the locus of $\mathbf{E}(0,t)$ for a plane wave with

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} \sin(\omega t + kz) + \hat{\mathbf{y}}\, 2\cos(\omega t + kz)$$

Determine the polarization state from your plot.

Solution:

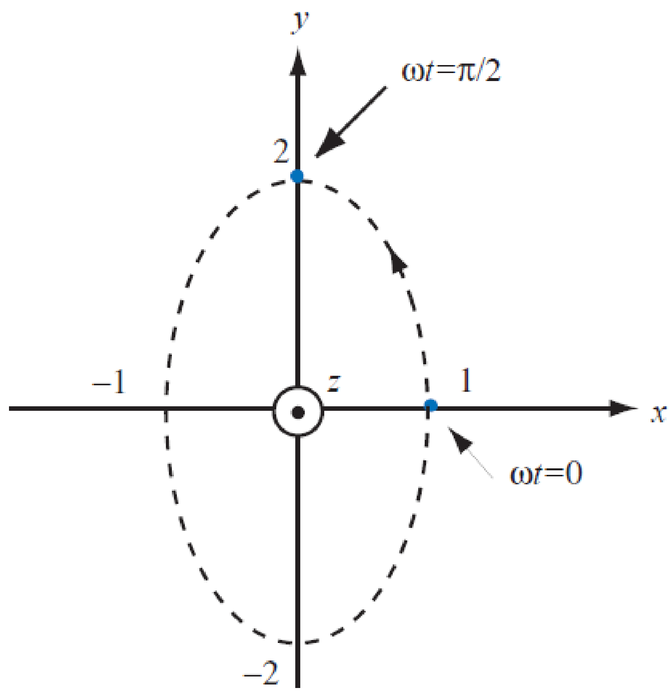

Figure P7.14: Locus of **E** versus time.

$$\mathbf{E} = \hat{\mathbf{x}} \sin(\omega t + kz) + \hat{\mathbf{y}}\, 2\cos(\omega t + kz).$$

Wave direction is $-\hat{\mathbf{z}}$. At $z = 0$,

$$\mathbf{E} = \hat{\mathbf{x}} \sin \omega t + \hat{\mathbf{y}}\, 2\cos \omega t.$$

Tip of **E** rotates in accordance with right hand (with thumb pointing along $-\hat{\mathbf{z}}$). Hence, wave state is RHE.

Problem 7.15 For each of the following combinations of parameters, determine if the material is a low-loss dielectric, a quasi-conductor, or a good conductor, and then calculate α, β, λ, u_p, and η_c:

(a) Glass with $\mu_r = 1$, $\varepsilon_r = 5$, and $\sigma = 10^{-12}$ S/m at 10 GHz.

(b) Animal tissue with $\mu_r = 1$, $\varepsilon_r = 12$, and $\sigma = 0.3$ S/m at 100 MHz.

(c) Wood with $\mu_r = 1$, $\varepsilon_r = 3$, and $\sigma = 10^{-4}$ S/m at 1 kHz.

Solution: Using equations given in Table 7-1:

	Case (a)	Case (b)	Case (c)
$\sigma/\omega\varepsilon$	3.6×10^{-13}	4.5	600
Type	low-loss dielectric	quasi-conductor	good conductor
α	8.42×10^{-11} Np/m	9.75 Np/m	6.3×10^{-4} Np/m
β	468.3 rad/m	12.16 rad/m	6.3×10^{-4} rad/m
λ	1.34 cm	51.69 cm	10 km
u_p	1.34×10^8 m/s	0.52×10^8 m/s	0.1×10^8 m/s
η_c	$\simeq 168.5\ \Omega$	$39.54 + j31.72\ \Omega$	$6.28(1+j)\ \Omega$

Problem 7.20 The skin depth of a certain nonmagnetic conducting material is 3 μm at 2 GHz. Determine the phase velocity in the material.

Solution: For a good conductor, $\alpha = \beta$, and for any material $\delta_s = 1/\alpha$. Hence,

$$u_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = 2\pi f \delta_s = 2\pi \times 5 \times 10^9 \times 3 \times 10^{-6} = 9.42 \times 10^4 \quad \text{(m/s)}.$$

Problem 7.22 The electric field of a plane wave propagating in a nonmagnetic medium is given by

$$\mathbf{E} = \hat{\mathbf{z}}25e^{-30x}\cos(2\pi \times 10^9 t - 40x) \quad \text{(V/m)}$$

Obtain the corresponding expression for $\mathbf{H}$.

Solution: From the given expression for $\mathbf{E}$,

$$\begin{aligned} \omega &= 2\pi \times 10^9 \quad \text{(rad/s)}, \\ \alpha &= 30 \quad \text{(Np/m)}, \\ \beta &= 40 \quad \text{(rad/m)}. \end{aligned}$$

From (7.65a) and (7.65b),

$$\alpha^2 - \beta^2 = -\omega^2 \mu \varepsilon' = -\omega^2 \mu_0 \varepsilon_0 \varepsilon_r' = -\frac{\omega^2}{c^2}\,\varepsilon_r',$$

$$2\alpha\beta = \omega^2 \mu \varepsilon'' = \frac{\omega^2}{c^2}\,\varepsilon_r''.$$

Using the above values for ω, α, and β, we obtain the following:

$$\begin{aligned}
\varepsilon_r' &= 1.6, \\
\varepsilon_r'' &= 5.47. \\
\eta_c &= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \\
&= \frac{\eta_0}{\sqrt{\varepsilon_r'}}\left(1 - j\frac{\varepsilon_r''}{\varepsilon_r'}\right)^{-1/2} = \frac{377}{\sqrt{1.6}}\left(1 - j\frac{5.47}{1.6}\right)^{-1/2} = 157.9\,e^{j36.85^\circ} \quad (\Omega). \\
\widetilde{\mathbf{E}} &= \hat{\mathbf{z}}25e^{-30x}e^{-j40x}, \\
\widetilde{\mathbf{H}} &= \frac{1}{\eta_c}\hat{\mathbf{k}} \times \widetilde{\mathbf{E}} = \frac{1}{157.9\,e^{j36.85^\circ}}\hat{\mathbf{x}} \times \hat{\mathbf{z}}25e^{-30x}e^{-j40x} = -\hat{\mathbf{y}}0.16e^{-30x}e^{-40x}e^{-j36.85^\circ}, \\
\mathbf{H} &= \Re\mathfrak{e}\{\widetilde{\mathbf{H}}e^{j\omega t}\} = -\hat{\mathbf{y}}0.16e^{-30x}\cos(2\pi \times 10^9 t - 40x - 36.85^\circ) \quad \text{(A/m)}.
\end{aligned}$$

Problem 7.24 In a nonmagnetic, lossy, dielectric medium, a 300-MHz plane wave is characterized by the magnetic field phasor

$$\widetilde{\mathbf{H}} = (\hat{\mathbf{x}} - j4\hat{\mathbf{z}})e^{-2y}e^{-j9y} \quad \text{(A/m)}$$

Obtain time-domain expressions for the electric and magnetic field vectors.

Solution:

$$\widetilde{\mathbf{E}} = -\eta_c \hat{\mathbf{k}} \times \widetilde{\mathbf{H}}.$$

To find η_c, we need ε' and ε''. From the given expression for $\widetilde{\mathbf{H}}$,

$$\alpha = 2 \quad \text{(Np/m)},$$
$$\beta = 9 \quad \text{(rad/m)}.$$

Also, we are given than $f = 300$ MHz $= 3 \times 10^8$ Hz. From (7.65a),

$$\alpha^2 - \beta^2 = -\omega^2 \mu \varepsilon',$$
$$4 - 81 = -(2\pi \times 3 \times 10^8)^2 \times 4\pi \times 10^{-7} \times \varepsilon_r' \times \frac{10^{-9}}{36\pi},$$

whose solution gives

$$\varepsilon_r' = 1.95.$$

Similarly, from (7.65b),

$$2\alpha\beta = \omega^2 \mu \varepsilon'',$$
$$2 \times 2 \times 9 = (2\pi \times 3 \times 10^8)^2 \times 4\pi \times 10^{-7} \times \varepsilon_r'' \times \frac{10^{-9}}{36\pi},$$

which gives

$$\varepsilon_r'' = 0.91.$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$
$$= \frac{\eta_0}{\sqrt{\varepsilon_r'}} \left(1 - j\frac{0.91}{1.95}\right)^{-1/2} = \frac{377}{\sqrt{1.95}}(0.93 + j0.21) = 256.9\, e^{j12.6^\circ}.$$

Hence,

$$\widetilde{\mathbf{E}} = -256.9\, e^{j12.6^\circ} \hat{\mathbf{y}} \times (\hat{\mathbf{x}} - j4\hat{\mathbf{z}})e^{-2y}e^{-j9y}$$
$$= (\hat{\mathbf{x}}\, j4 + \hat{\mathbf{z}})\, 256.9\, e^{-2y}e^{-j9y}e^{j12.6^\circ}$$
$$= (\hat{\mathbf{x}}\, 4e^{j\pi/2} + \hat{\mathbf{z}})\, 256.9\, e^{-2y}e^{-j9y}e^{j12.6^\circ},$$

$$\begin{aligned}
\mathbf{E} &= \Re\mathfrak{e}\{\widetilde{\mathbf{E}}e^{j\omega t}\} \\
&= \hat{\mathbf{x}}\,1.03\times 10^3 e^{-2y}\cos(\omega t - 9y + 102.6^\circ) \\
&\quad + \hat{\mathbf{z}}\,256.9\,e^{-2y}\cos(\omega t - 9y + 12.6^\circ) \quad \text{(V/m)}, \\
\mathbf{H} &= \Re\mathfrak{e}\{\widetilde{\mathbf{H}}e^{j\omega t}\} \\
&= \Re\mathfrak{e}\{(\hat{\mathbf{x}} + j4\hat{\mathbf{z}})e^{-2y}e^{-j9y}e^{j\omega t}\} \\
&= \hat{\mathbf{x}}e^{-2y}\cos(\omega t - 9y) + \hat{\mathbf{z}}\,4e^{-2y}\sin(\omega t - 9y) \quad \text{(A/m)}.
\end{aligned}$$

Problem 7.27 The magnetic field of a plane wave traveling in air is given by $\mathbf{H} = \hat{\mathbf{x}}\,50\sin(2\pi\times 10^7 t - ky)$ (mA/m). Determine the average power density carried by the wave.

Solution:

$$\begin{aligned}
\mathbf{H} &= \hat{\mathbf{x}}50\sin(2\pi\times 10^7 t - ky) \quad \text{(mA/m)}, \\
\mathbf{E} &= -\eta_0\hat{\mathbf{y}}\times\mathbf{H} = \hat{\mathbf{z}}\eta_0 50\sin(2\pi\times 10^7 t - ky) \quad \text{(mV/m)}, \\
\mathbf{S}_{\text{av}} &= (\hat{\mathbf{z}}\times\hat{\mathbf{x}})\frac{\eta_0(50)^2}{2}\times 10^{-6} = \hat{\mathbf{y}}\frac{120\pi}{2}(50)^2\times 10^{-6} = \hat{\mathbf{y}}0.48 \quad (\text{W/m}^2).
\end{aligned}$$

Problem 7.28 A wave traveling in a nonmagnetic medium with $\varepsilon_r = 9$ is characterized by an electric field given by

$$\begin{aligned}
\mathbf{E} = [&\hat{\mathbf{y}}\,3\cos(\pi\times 10^7 t + kx) \\
&- \hat{\mathbf{z}}\,2\cos(\pi\times 10^7 t + kx)] \quad \text{(V/m)}
\end{aligned}$$

Determine the direction of wave travel and average power density carried by the wave.

Solution:

$$\eta \simeq \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{120\pi}{\sqrt{9}} = 40\pi \quad (\Omega).$$

The wave is traveling in the negative x-direction.

$$\mathbf{S}_{\text{av}} = -\hat{\mathbf{x}}\frac{[3^2+2^2]}{2\eta} = -\hat{\mathbf{x}}\frac{13}{2\times 40\pi} = -\hat{\mathbf{x}}0.05 \quad (\text{W/m}^2).$$

Problem 7.32 At microwave frequencies, the power density considered safe for human exposure is 1 (mW/cm^2). A radar radiates a wave with an electric field amplitude E that decays with distance as $E(R) = (3{,}000/R)$ (V/m), where R is the distance in meters. What is the radius of the unsafe region?

Solution:

$$S_{\text{av}} = \frac{|E(R)|^2}{2\eta_0}, \qquad 1\ (\text{mW/cm}^2) = 10^{-3}\ \text{W/cm}^2 = 10\ \text{W/m}^2,$$

$$10 = \left(\frac{3\times10^3}{R}\right)^2 \times \frac{1}{2\times120\pi} = \frac{1.2\times10^4}{R^2},$$

$$R = \left(\frac{1.2\times10^4}{10}\right)^{1/2} = 34.64\ \text{m}.$$

Problem 7.36 A team of scientists is designing a radar as a probe for measuring the depth of the ice layer over the antarctic land mass. In order to measure a detectable echo due to the reflection by the ice-rock boundary, the thickness of the ice sheet should not exceed three skin depths. If $\varepsilon_r' = 3$ and $\varepsilon_r'' = 10^{-2}$ for ice and if the maximum anticipated ice thickness in the area under exploration is 1.2 km, what frequency range is useable with the radar?

Solution:

$$3\delta_s = 1.2\ \text{km} = 1200\ \text{m}$$
$$\delta_s = 400\ \text{m}.$$

Hence,

$$\alpha = \frac{1}{\delta_s} = \frac{1}{400} = 2.5\times10^{-3} \quad (\text{Np/m}).$$

Since $\varepsilon''/\varepsilon' \ll 1$, we can use (7.75a) for α:

$$\alpha = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{2\pi f\varepsilon_r''\varepsilon_0}{2\sqrt{\varepsilon_r'}\sqrt{\varepsilon_0}}\sqrt{\mu_0} = \frac{\pi f\varepsilon_r''}{c\sqrt{\varepsilon_r}} = \frac{\pi f\times10^{-2}}{3\times10^8\sqrt{3}} = 6f\times10^{-11}\text{Np/m}.$$

For $\alpha = 2.5\times10^{-3} = 6f\times10^{-11}$,

$$f = 41.6\ \text{MHz}.$$

Since α increases with increasing frequency, the useable frequency range is

$$f \le 41.6\ \text{MHz}.$$